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STUDY PACKAGE

Subject : Mathematics

Topic : **Idefinite & Definite Integration**

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Indefinite Integration

1. If f & g are functions of x such that $g'(x) = f(x)$ then,

$\int f(x)\,dx = g(x) + c \Leftrightarrow \frac{d}{dx}\{g(x)+c\} = f(x)$, where c is called the **constant of integration**.

2. Standard Formula:

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c,\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ln(ax+b) + c$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$

(iv) $\int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{\ell n\, a} + c;\ a > 0$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)\,dx = \frac{1}{a}\ln\sec(ax+b) + c$

(viii) $\int \cot(ax+b)\,dx = \frac{1}{a}\ln\sin(ax+b) + c$

(ix) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \text{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \sec(ax+b).\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + c$

(xii) $\int \text{cosec}(ax+b).\cot(ax+b)\,dx = -\frac{1}{a}\text{cosec}(ax+b) + c$

(xiii) $\int \sec x\,dx = \ln(\sec x + \tan x) + c$ **OR** $\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) + c$

(xiv) $\int \text{cosec}\,x\,dx = \ln(\text{cosec}\,x - \cot x) + c$ **OR** $\ln\tan\frac{x}{2} + c$ **OR** $-\ln(\text{cosec}\,x + \cot x) + c$

(xv) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

(xvi) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(xvii) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xviii) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left[x + \sqrt{x^2+a^2}\right]$ **OR** $\sinh^{-1}\frac{x}{a} + c$

(xix) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left[x + \sqrt{x^2-a^2}\right]$ **OR** $\cosh^{-1}\frac{x}{a} + c$

(xx) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

(xxi) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

(xxii) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(xxiii) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ell n\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) + c$

(xxiv) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ell n\left(\frac{x+\sqrt{x^2-a^2}}{a}\right) + c$

(xxv) $\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

(xxvi) $\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

3. Theorems on integration

(i) $\int c\,f(x).dx = c\int f(x).dx$

(ii) $\int (f(x) \pm g(x))dx = \int f(x)dx \pm g(x)dx$

(iii) $\int f(x)dx = g(x) + c \Rightarrow \int f(ax+b)dx = \frac{g(ax+b)}{a} + c$

Note : (i) every contineous function is integrable
(ii) the integral of a function reffered only by a constant.

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$\int f(x).dx = g(x) + c$

$= h(x) + c$

$g'(x) = f(x)$ & $h'(x) = f(x)$

$g'(x) - h'(x) = 0$

means, $g(x) - h(x) = c$

Example : Evaluate : $\int 4x^5\, dx$

Solution. $\int 4x^5\, dx = \frac{4}{6}\, x^6 + C = \frac{2}{3}\, x^6 + C.$

Example : Evaluate : $\int \left(x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right) dx$

Solution. $\int \left(x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right) dx$

$= \int x^3\, dx + \int 5x^2\, dx - \int 4dx + \int \frac{7}{x} dx + \int \frac{2}{\sqrt{x}} dx$

$= \int x^3\, dx + 5 . \int x^2\, dx - 4 . \int 1. dx + 7 . \int \frac{1}{x} dx + 2 . \int x^{-1/2} dx$

$= \frac{x^4}{4} + 5 . \frac{x^3}{3} - 4x + 7 \log |x| + 2 \left(\frac{x^{1/2}}{1/2}\right) + C$

$= \frac{x^4}{4} + \frac{5}{3} x^3 - 4x + 7 \log |x| + 4\sqrt{x} + C$

Example : Evaluate : $\int e^{x\log a} + e^{a\log x} + e^{a\log a}\, dx$

Solution. We have,

$\int e^{x\log a} + e^{a\log x} + e^{a\log a}\, dx$

$= \int e^{\log a^x} + e^{\log x^a} + e^{\log a^a}\, dx$ $= \int (a^x + x^a + a^a)\, dx$

$= \int a^x\, dx + \int x^a\, dx + \int a^a\, dx$ $= \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + a^a . x + C.$

Example : Evaluate : $\int \frac{2^x + 3^x}{5^x}\, dx$

Solution. $\int \frac{2^x + 3^x}{5^x}\, dx$

$= \int \left(\frac{2^x}{5^x} + \frac{3^x}{5^x}\right) dx$ $= \int \left[\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right] dx = \frac{(2/5)^x}{\log_e 2/5} + \frac{(3/5)^x}{\log_e 3/5} + C$

Example: Evaluate : $\int \sin^3 x \cos^3 x\, dx$

Solution. $= \frac{1}{8} \int (2\sin x \cos x)^3\, dx$

$= \frac{1}{8} \int \sin^3 2x\, dx$ $= \frac{1}{8} \int \frac{3\sin 2x - \sin 6x}{4}\, dx$

$= \frac{1}{32} \int (3\sin 2x - \sin 6x)\, dx$ $= \frac{1}{32}\left[-\frac{3}{2}\cos 2x + \frac{1}{6}\cos 6x\right] + C$

Example : Evaluate : $\int \frac{x^4}{x^2+1}\, dx$

Solution. $\int \frac{x^4}{x^2+1}\, dx$

$= \int \frac{x^4 - 1 + 1}{x^2+1}\, dx = \int \frac{x^4 - 1}{x^2+1} + \frac{1}{x^2+1}\, dx = \int (x^2 - 1)\, dx + \int \frac{1}{x^2+1}\, dx = \frac{x^3}{3} - x + \tan^{-1} x + C$

Example: Evaluate : $\int \frac{1}{4+9x^2}\, dx$

Solution. We have $\int \frac{1}{4+9x^2}$

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$= \frac{1}{9} \int \frac{1}{\frac{4}{9} + x^2} dx$

$= \frac{1}{9} \int \frac{1}{(2/3)^2 + x^2} dx \qquad = \frac{1}{9} \cdot \frac{1}{(2/3)} \tan^{-1}\left(\frac{x}{2/3}\right) + C = \frac{1}{6} \tan^{-1}\left(\frac{3x}{2}\right) + C$

Example : $\int \cos x \cos 2x \, dx$

Solution. $\int \cos x \cos 2x \, dx$

$= \frac{1}{2} \int 2\cos x \cos 2x \, dx$

$= \frac{1}{2} \int (\cos 3x + \cos x) \, dx \qquad = \frac{1}{2} \left(\frac{\sin 3x}{3} + \frac{\sin x}{1}\right) + c$

Self Practice Problems

1. Evaluate : $\int \tan^2 x \, dx$ **Ans.** $\tan x - x + C$

2. Evaluate : $\int \frac{1}{1+\sin x} dx$ **Ans.** $\tan x - \sec x + C$

4. Integration by Subsitutions

If we subsitute $x = \phi(t)$ in a integral then
(i) everywhere x will be replaced in terms of t. (ii) dx also gets converted in terms of dt.
(iii) $\phi(t)$ should be able to take all possible value that x can take.

Example : Evaluate : $\int x^3 \sin x^4 \, dx$

Solution. We have

$I = \int x^3 \sin x^4 \, dx$

Let $x^4 = t \Rightarrow d(x^4) = dt \Rightarrow 4x^3 dx = dt \Rightarrow dx = \frac{1}{4x^3} dt$

Example : $\int \frac{(\ell n\, x)^2}{x} dx$

Solution. $\int \frac{(\ell n\, x)^2}{x} dx$

Put $\ell nx = t \Rightarrow \frac{1}{x} dx = dt$

$= \int t^2 . \left(\frac{dx}{x}\right) \qquad = \int t^2 dt$

$= \frac{t^3}{3} + c \qquad = \frac{(\ell n x)^3}{3} + c$

Example : Evaluate $\int (1 + \sin^2 x) \cos x \, dx$

Solution. Put $\sin x = t$
$\cos x \, dx = dt$

$\int (1+t^2)\, dt = t + \frac{t^3}{3} + c \qquad = \sin x + \frac{\sin^3 x}{3} + c$

Example : Evaluate : $\int \frac{x}{x^4 + x^2 + 1} dx$

Solution. We have,

$I = \int \frac{x}{x^4 + x^2 + 1} dx = \int \frac{x}{(x^2)^2 + x^2 + 1} dx$

Let $x^2 = t$, then, $d(x^2) = dt \Rightarrow 2x\, dx = dt \Rightarrow dx = \frac{dt}{2x}$

$I = \int \frac{x}{t^2 + t + 1} . \frac{dt}{2x} \qquad = \frac{1}{2} \int \frac{1}{t^2 + t + 1} dt \qquad = \frac{1}{2} \int \frac{1}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dt$

$= \frac{1}{2} . \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$

$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C.$

Note: **(i)** $\int [f(x)]^n f'(x)\, dx = \frac{(f(x))^{n+1}}{n+1}$ **(ii)** $\int \frac{f'(x)}{[f(x)]^n} dx = \frac{(f(x))^{1-n}}{1-n}$

(iii) $\int \frac{dx}{x(x^n+1)}$ $n \in N$ Take x^n common & put $1 + x^{-n} = t$.

(iv) $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}$ $n \in N$, take x^n common & put $1+x^{-n} = t^n$

(v) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$ take x^n common as x and put $1 + x^{-n} = t$.

Self Practice Problems

1. $\int \frac{\sec^2 x}{1+\tan x} dx$ **Ans.** $\ell n\,|1 + \tan x| + C$

2. $\int \frac{\sin(\ell n x)}{x} dx$ **Ans.** $-\cos(\ell n\, x) + C$

5. Integration by Part :

$\int (f(x)\, g(x))\, dx = f(x) \int (g(x))\, dx - \int \left(\frac{d}{dx}(f(x)) \int (g(x))\, dx\right) dx$

(i) when you find integral $\int g(x)\, dx$ then it will not contain arbitarary constant.

(ii) $\int g(x)\, dx$ should be taken as same both terms.

(iii) the choice of f(x) and g(x) is decided by ILATE rule.
the function will come later is taken an integral function.

I → Inverse function
L → Logrithimic function
A → Algeberic function
T → Trigonometric function
E → Exponential function

Example : Evaluate : $\int x \tan^{-1} x\, dx$

Solution. $\int x \tan^{-1} x\, dx$

$= (\tan^{-1} x)\, \frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2}\, dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2+1-1}{x^2+1} dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int 1 - \frac{1}{x^2+1} dx$

$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + C.$

Example : Evaluate : $\int x \log(1+x)\, dx$

Solution. $\int x \log(1+x)\, dx$

$= \log(x+1) \cdot \frac{x^2}{2} - \int \frac{1}{x+1} \cdot \frac{x^2}{2}\, dx$

$= \frac{x^2}{2} \log(x+1) - \frac{1}{2} \int \frac{x^2}{x+1} dx = \frac{x^2}{2} \log(x+1) - \frac{1}{2} \int \frac{x^2-1+1}{x+1} dx$

$= \frac{x^2}{2} \log(x+1) - \frac{1}{2} \int \frac{x^2-1}{x+1} + \frac{1}{x+1} dx$

$= \frac{x^2}{2} \log(x+1) - \frac{1}{2} \left[\int \left((x-1) + \frac{1}{x+1}\right) dx\right]$

$= \frac{x^2}{2} \log(x+1) - \frac{1}{2} \left[\frac{x^2}{2} - x + \log|x+1|\right] + C$

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Example : Evaluate : $\int e^{2x}\sin 3x\ dx$

Solution. Let $I = \int e^{2x}\sin 3x\ dx$. Then,

$I = \int e^{2x}\sin 3x\ dx$

$\Rightarrow\ I = e^{2x}\left(-\frac{\cos 3x}{3}\right) - \int 2e^{2x}\left(-\frac{\cos 3x}{3}\right)dx \quad \Rightarrow \quad I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x\ dx$

$\Rightarrow\ I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left[e^{2x}\frac{\sin 3x}{3} - \int 2e^{2x}\frac{\sin 3x}{3}dx\right]$

$\Rightarrow\ I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9}\int e^{2x}\sin 3x\ dx$

$\Rightarrow\ I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9}I \quad \Rightarrow \quad I + \frac{4}{9}I = \frac{e^{2x}}{9}(2\sin 3x - 3\cos 3x)$

$\Rightarrow\ \frac{13}{9}I = \frac{e^{2x}}{9}(2\sin 3x - 3\cos 3x) \quad \Rightarrow \quad I = \frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x) + C$

Note : (i) $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + c$ **(ii)** $\int [f(x) + xf'(x)]\,dx = x f(x) + c$

Example : $\int e^x \frac{x}{(x+1)^2}dx$

Solution. $\int e^x \frac{x+1-1}{(x+1)^2}dx \quad \Rightarrow \quad \int e^x\left(\frac{1}{(x+1)} - \frac{1}{(x+1)^2}\right)dx \quad = \frac{e^x}{(x+1)} + c$

Example : $\int e^x\left(\frac{1-\sin x}{1-\cos x}\right)dx$

Solution. $\int e^x\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right)dx$

$\Rightarrow\ \int e^x\left(\frac{1}{2}\text{cosec}^2 - \cot\frac{x}{2}\right)dx \quad = -e^x\cot\frac{x}{2} + c$

Example : $\int\left[\ell n(\ell nx) + \frac{1}{(\ell nx)^2}\right]dx$

Solution. put $x = e^t$

$\Rightarrow\ \int e^t\left(\ell nt + \frac{1}{t^2}\right)dt \quad \Rightarrow \quad \int e^t\left(\ell nt - \frac{1}{t} + \frac{1}{t} + \frac{1}{t^2}\right)dt \quad = e^t\left(\ell nt - \frac{1}{t}\right) + c$

$\Rightarrow\ x\left[\ell n(\ell nx) - \frac{1}{\ell nx}\right] + c$

Self Practice Problems

1. $\int x\sin x\,dx$ **Ans.** $-x\cos x + \sin x + C$
2. $\int x^2e^x\,dx$ **Ans.** $x^2e^x - 2xe^x + 2e^x + C$

6. Integration of Rational Algebraic Functions by using Partial Fractions:

PARTIAL FRACTIONS :

If f(x) and g(x) are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function of a rational function of x.

If degree of f(x) < degree of g(x), then $\frac{f(x)}{g(x)}$ is called a proper rational function.

If degree of f(x) ≥ degree of g(x) then $\frac{f(x)}{g(x)}$ is called an improper rational function

If $\frac{f(x)}{g(x)}$ is an improper rational function, we divide f(x) by g(x) so that the rational function $\frac{f(x)}{g(x)}$ is expressed in the form $\phi(x) + \frac{\Psi(x)}{g(x)}$ where $\phi(x)$ and $\psi(x)$ are polynomials such that the degree of $\psi(x)$ is less than that of g(x). Thus, $\frac{f(x)}{g(x)}$ is expressible as the sum of a polynomial and a proper rational function.

Any proper rational function $\frac{f(x)}{g(x)}$ can be expressed as the sum of rational functions, each having a simple factor of g(x). Each such fraction is called a partial fraction and the process of obtained them is called the resolutions or decomposition of $\frac{f(x)}{g(x)}$ into partial fractions.

The resolution of $\frac{f(x)}{g(x)}$ into partial fractions depends mainly upon the nature of the factors of g(x) as discussed below.

CASE I When denominator is expressible as the product of non-repeating linear factors.

Let $g(x) = (x - a_1)(x - a_2) \ldots(x - a_n)$. Then, we assume that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots + \frac{A_n}{x-a_n}$$

where $A_1, A_2, \ldots A_n$ are constants and can be determined by equating the numerator on R.H.S. to the numerator on L.H.S. and then substituting $x = a_1, a_2, \ldots, a_n$.

Example : Resolve $\frac{3x+2}{x^3-6x^2+11x-6}$ into partial fractions.

Solution. We have, $\frac{3x+2}{x^3-6x^2+11x-6} = \frac{3x+2}{(x-1)(x-2)(x-3)}$

Let $\frac{3x+2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{B}{x-3}$. Then,

$\Rightarrow \frac{3x+2}{(x-1)(x-2)(x-3)} = \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$

$\Rightarrow 3x + 2 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$ ....(i)

Putting x – 1 = 0 or x = 1 in (i), we get

$5 = A(1 - 2)(1 - 3) \Rightarrow A = \frac{5}{2}$,

Putting x – 2 = 0 or, x = 2 in (i), we obtain

$8 = B(2 - 1)(2 - 3) \Rightarrow B = -8$.

Putting x – 3 = 0 or, x = 3 in (i), we obtain

$11 = C(3 - 1)(3 - 2) \Rightarrow C = \frac{11}{2}$.

$\therefore \frac{3x+2}{x^3-6x^2+11x-6} = \frac{3x+2}{(x-1)(x-2)(x-3)} = \frac{5}{2(x-1)} - \frac{8}{x-2} + \frac{11}{2(x-3)}$

Note : In order to determine the value of constants in the numerator of the partial fraction corresponding to the non-repeated linear factor px + q in the denominator of a rational expression, we may proceed as follows :

Replace $x = -\frac{q}{p}$ (obtained by putting px + q = 0) everywhere in the given rational expression except in the factor px + q itself. For example, in the above illustration the value of A is obtained by replacing x by 1 in all factors of $\frac{3x+2}{(x-1)(x-2)(x-3)}$ except (x – 1) i.e.

$A = \frac{3\times1+2}{(1-2)(1-3)} = \frac{5}{2}$

Similarly, we have

$B = \frac{3\times2+1}{(1-2)(2-3)} = -8$ and, $C = \frac{3\times3+2}{(3-1)(3-2)} = \frac{11}{2}$

Example : Resolve $\frac{x^3-6x^2+10x-2}{x^2-5x+6}$ into partial fractions.

Solution. Here the given function is an improper rational function. On dividing we get

$\frac{x^3-6x^2+10x-2}{x^2-5x+6} = x - 1 + \frac{(-x+4)}{(x^2-5x+6)}$ ....(i)

we have, $\frac{-x+4}{x^2-5x+6} = \frac{-x+4}{(x-2)(x-3)}$

So, let $\frac{-x+4}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$ – x + 4 = A(x – 3) + B(x – 2) ....(ii)

Putting x – 3 = 0 or, x = 3 in (ii), we get

$1 = B(1) \Rightarrow B = 1$.

Putting x – 2 = 0 or, x = 2 in (ii), we get

$2 = A(2 - 3) \Rightarrow A = -2$

$\therefore \frac{-x+4}{(x-2)(x-3)} = \frac{-2}{x-2} + \frac{1}{x-3}$ Hence $\frac{x^3-6x^2+10x-2}{x^2-5x+6} = x - 1 - \frac{2}{x-2} + \frac{2}{x-3}$

CASE II When the denominator g(x) is expressible as the product of the linear factors such that some of them are repeating.

Example $\frac{1}{g(x)} = \frac{1}{(x-a)^k(x-a_1)(x-a_2).......(x-a_r)}$ this can be expressed as

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + ...+ \frac{A_k}{(x-a)^k} + \frac{B_1}{(x-a_1)} + \frac{B_2}{(x-a_2)} + ... + \frac{B_r}{(x-a_r)}$$

Now to determine constants we equate numerators on both sides. Some of the constants are determined by substitution as in case I and remaining are obtained by

The following example illustrate the procedure.

Example : Resolve $\frac{3x-2}{(x-1)^2(x+1)(x+2)}$ into partial fractions, and evaluate $\int \frac{(3x-2)dx}{(x-1)^2(x+1)(x+2)}$

Solution. Let $\frac{3x-2}{(x-1)^2(x+1)(x+2)} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{x+1} + \frac{A_4}{x+2}$

$\Rightarrow$ $3x - 2 = A_1 (x - 1) (x + 1) (x + 2) + A_2 (x + 1) (x + 2) + A_3 (x - 1)^2 (x + 2) + A_4 (x - 1)^2 (x + 1)$ ....(i)

Putting x – 1 = 0 or, x = 1 in (i) we get

$1 = A_2 (1 + 1) (1 + 2) \Rightarrow A_2 = \frac{1}{6}$

Putting x + 1 = 0 or, x = –1 in (i) we get

$-5 = A_3 (-2)^2 (-1 + 2) \Rightarrow A_3 = -\frac{5}{4}$

Putting x + 2 = 0 or, x = –2 in (i) we get

$-8 = A_4 (-3)^2 (-1) \Rightarrow A_4 = \frac{8}{9}$

Now equating coefficient of x^3 on both sides, we get $0 = A_1 + A_3 + A_4$

$\Rightarrow$ $A_1 = -A_3 - A_4 = \frac{5}{4} - \frac{8}{9} = \frac{13}{36}$

$\therefore$ $\frac{3x-2}{(x-1)^2(x+1)(x+2)} = \frac{13}{36(x-1)} + \frac{1}{6(x-1)^2} - \frac{5}{4(x+1)} + \frac{8}{9(x+2)}$

and hence $\int \frac{(3x-2)dx}{(x-1)^2(x+1)(x+2)}$

$= \frac{13}{36} \ell n |x - 1| - \frac{1}{6(x-1)} - \frac{5}{4} \ell n |x + 1| + \frac{8}{9} \ell n |x + 2| + c$

CASE III When some of the factors of denominator g(x) are quadratic but non-repeating. Corresponding to each quadratic factor $ax^2 + bx + c$, we assume partial fraction of the type $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of similar powers of x in the numerator of both sides.

In practice it is advisable to assume partial fractions of the type $\frac{A(2ax+b)}{ax^2+bx+c} + \frac{B}{ax^2+bx+c}$

The following example illustrates the procedure

Example : Resolve $\frac{2x-1}{(x+1)(x^2+2)}$ into partial fractions and evaluate $\int \frac{2x-1}{(x+1)(x^2+2)} dx$

Solution. Let $\frac{2x-1}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2}$. Then,

$\frac{2x-1}{(x+1)(x^2+2)} = \frac{A(x^2+2)+(Bx+C)(x+1)}{(x+1)(x^2+2)}$

$\Rightarrow$ $2x - 1 = A (x^2 + 2) + (Bx + C) (x + 1)$ ...(i)

Putting x + 1 = 0 or, x = –1 in (i), we get – 3 = A(3) $\Rightarrow$ A = –1.

Comparing coefficients of the like powers of x on both sides of (i), we get

A + B = 0, C + 2A = –1 and C + B = 2

$\therefore$ –1 + B = 0, C – 2 = –1 (Putting A = –1)

$\Rightarrow$ B = 1, C = 1 $\therefore$ $\frac{2x-1}{(x+1)(x^2+2)} = -\frac{1}{x+1} + \frac{x+1}{x^2+2}$

Hence $\int \frac{2x-1}{(x+1)(x^2+2)} dx = -\ell n |x + 1| + \frac{1}{2} \ell n |x^2 + 1| + \frac{1}{\sqrt{2}} \tan^{-1} \frac{x}{\sqrt{2}} + c$

CASE IV When some of the factors of the denominator g(x) are quadratic and repeating fractions of the form $\left\{\frac{A_0(2ax+b)}{ax^2+bx+c} + \frac{A_1}{ax^2+bx+c}\right\} + \left\{\frac{A_1(2ax+b)}{(ax^2+bx+c)^2} + \frac{A_2}{(ax^2+bx+c)^2}\right\}$

$+ \ldots\ldots. + \left\{\frac{A_{2k-1}(2ax+b)}{(ax^2+bx+c)^k} + \frac{A_{2k}}{(ax^2+bx+c)^k}\right\}$

The following example illustrates the procedure.

Example: Resolve $\frac{2x-3}{(x-1)(x^2+1)^2}$ into partial fractions.

Solution. Let $\frac{2x-3}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$. Then,

$2x - 3 = A(x^2 + 1)^2 + (Bx + C)(x - 1)(x^2 + 1) + (Dx + E)(x - 1)$ ......(i)

Putting x = 1 in (i), we get $-1 = A(1 + 1)^2 \Rightarrow A = -\frac{1}{4}$

Equation coefficients of like powers of x, we have

A + B = 0, C – B = 0, 2A + B – C + D = 0, C + E – B – D = 2 and A – C – E = –3.

Putting $A = -\frac{1}{4}$ and solving these equations, we get

$B = \frac{1}{4} = C,\ D = \frac{1}{2}$ and $E = \frac{5}{2}$

$\therefore \quad \frac{2x-3}{(x-1)(x^2+1)^2} = \frac{-1}{4(x-1)} + \frac{x+1}{4(x^2+1)} + \frac{x+5}{2(x^2+1)^2}$

Example : Resolve $\frac{2x}{x^3-1}$ into partial fractions.

Solution. We have, $\frac{2x}{x^3-1} = \frac{2x}{(x-1)(x^2+x+1)}$

So, let $\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$. Then,

$2x = A(x^2 + x + 1) + (Bx + C)(x - 1)$ ....(i)

Putting x – 1 = 0 or, x = 1 in (i), we get $2 = 3A \Rightarrow A = \frac{2}{3}$

Putting x = 0 in (i), we get $A - C = 0 \Rightarrow C = A = \frac{2}{3}$

Putting x = – 1 in (i), we get –2 = A + 2B – 2 C.

$\Rightarrow \quad -2 = \frac{2}{3} + 2B - \frac{4}{3} \Rightarrow B = -\frac{2}{3}$

$\therefore \quad \frac{2x}{x^3-1} = \frac{2}{3}\cdot\frac{1}{x-1} + \frac{-2/3x+2/3}{x^2+x+1}$ or, $\frac{2x}{x^3-1} = \frac{2}{3}\frac{1}{x-1} + \frac{2}{3}\frac{1-x}{x^2+x+1}$

Self Practice Problems

1. (i) $\int \frac{1}{(x+2)(x+3)}dx$ **Ans.** $\ell n\left|\frac{x+2}{x+3}\right| + C$

(ii) $\int \frac{dx}{(x+1)(x^2+1)}$ **Ans.** $\frac{1}{2}\ \ell n\,|x+1| - \frac{1}{4}\ \ell n\,(x^2+1) + \frac{1}{2}\tan^{-1}(x) + C$

7. Integration of type $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\ dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

Example : Evaluate : $\int \sqrt{x^2+2x+5}\ dx$

Solution. We have,

$\int \sqrt{x^2+2x+5} = \int \sqrt{x^2+2x+1+4}\ dx$

$= \frac{1}{2}(x+1)\sqrt{(x-1)^2+2^2} + \frac{1}{2}.(2)^2 \log|(x+1) + \sqrt{(x+1)^2+2^2}| + C$

$= \frac{1}{2}(x+1)\sqrt{x^2-2x+5} + 2\log|(x+1) + \sqrt{x^2+2x+5}| + C$

Example : Evaluate : $\int \frac{1}{x^2-x+1}dx$

Solution. $\int \frac{1}{x^2-x+1}dx = \int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+1}dx = \int \frac{1}{(x-1/2)^2+3/4}dx$

$= \int \frac{1}{(x-1/2)^2+(\sqrt{3}/2)^2}dx = \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-1/2}{\sqrt{3}/2}\right) + C$

$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C.$

Example : Evaluate : $\int \frac{1}{\sqrt{9+8x-x^2}}\,dx$

Solution. $\int \frac{1}{\sqrt{9+8x-x^2}}\,dx$

$= \int \frac{1}{\sqrt{-\{x^2-8x-9\}}}\,dx \qquad = \int \frac{1}{\sqrt{-\{x^2-8x+16-25\}}}\,dx$

$= \int \frac{1}{-\{(x-4)^2-5^2\}}\,dx = \int \frac{1}{\sqrt{5^2-(x-4)^2}}\,dx = \sin^{-1}\left(\frac{x-4}{5}\right) + C$

Self Practice Problems

1. $\int \frac{1}{2x^2+x-1}\,dx$ **Ans.** $\frac{1}{3}\,\ell n\left|\frac{2x-1}{2x+2}\right| + C$

2. $\int \frac{1}{\sqrt{2x^2+3x-2}}\,dx$ **Ans.** $\frac{1}{\sqrt{2}}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x-1}\right| + C$

8. Integration of type

$$\int \frac{px+q}{ax^2+bx+c}\,dx,\ \int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx,\ \int (px+q)\sqrt{ax^2+bx+c}\,dx$$

Express px + q = A (differential co–efficient of denominator) + B.

Example : Evaluate : $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx$

Solution. $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx$

$= \int \frac{(2x+4)-1}{\sqrt{x^2+4x+1}}\,dx \qquad = \int \frac{2x+4}{\sqrt{x^2+4x+1}}\,dx - \int \frac{1}{\sqrt{x^2+4x+1}}\,dx$

$= \int \frac{dt}{\sqrt{t}} - \int \frac{1}{\sqrt{(x+2)^2-(\sqrt{3})^2}}\,dx$, where $t = x^2 + 4x + 1$

$= 2\sqrt{t} - \log|(x+2) + \sqrt{x^2+4x+1}| + C$

$= 2\sqrt{x^2+4x+1} - \log|x+2+\sqrt{x^2+4x+1}| + C$

Example : Evaluate : $\int (x-5)\sqrt{x^2+x}\,dx$

Solution. Let $(x-5) = \lambda \cdot \frac{d}{dx}(x^2+x) + \mu$. Then,

$x - 5 = \lambda(2x+1) + \mu$.

Comparing coefficients of like powers of x, we get

$1 = 2\lambda$ and $\lambda + \mu = -5 \Rightarrow \lambda = \frac{1}{2}$ and $\mu = -\frac{11}{2}$

$\int (x-5)\sqrt{x^2+x}\,dx$

$= \int \left(\frac{1}{2}(2x+1) - \frac{11}{2}\right)\sqrt{x^2+x}\,dx$

$= \int \frac{1}{2}(2x+1)\sqrt{x^2+x}\,dx - \frac{11}{2}\int \sqrt{x^2+x}\,dx$

$= \frac{1}{2}\int (2x+1)\sqrt{x^2+x}\,dx - \frac{11}{2}\int \sqrt{x^2+x}\,dx$

$= \frac{1}{2}\int \sqrt{t}\,dt - \frac{11}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\,dx$ where $t = x^2 + x$

$= \frac{1}{2} \cdot \frac{t^{3/2}}{3/2} - \frac{11}{2}\left[\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right.\right]$

$$- \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 \log\left[\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right] + C$$

$$= \frac{1}{3} t^{3/2} - \frac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\ell n\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right] + C$$

$$= \frac{1}{3}(x^2+x)^{3/2} - \frac{11}{2}\left[\frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\ell n\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right] + C$$

Self Practice Problems

1. $\int \frac{x+1}{x^2+x+3}\,dx$ **Ans.** $\frac{1}{2}\log|x^2+x+3| + \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{2x+1}{\sqrt{11}}\right) + C$

2. $\int \frac{6x-5}{\sqrt{3x^2-5x+1}}\,dx$ **Ans.** $2\sqrt{3x^2-5x+1} + C$

3. $\int (x-1)\sqrt{1+x+x^2}\,dx$

Ans. $\frac{1}{3}(x^2+x+1)^{3/2} - \frac{3}{8}(2x+1)\sqrt{1+x+x^2} - \frac{9}{16}\log\left(2x+1+2\sqrt{x^2+x+1}\right) + C$

9. Integration of trigonometric functions

(i) $\int \frac{dx}{a+b\sin^2 x}$ OR $\int \frac{dx}{a+b\cos^2 x}$ OR $\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$

Multiply Nr & Dr by $\sec^2 x$ & put $\tan x = t$.

(ii) $\int \frac{dx}{a+b\sin x}$ OR $\int \frac{dx}{a+b\cos x}$ OR $\int \frac{dx}{a+b\sin x + c\cos x}$

Hint: Convert sines & cosines into their respective tangents of half the angles and then, put $\tan\frac{x}{2} = t$

(iii) $\int \frac{a.\cos x + b.\sin x + c}{\ell.\cos x + m.\sin x + n}\,dx$. Express Nr $\equiv$ A(Dr) + B $\frac{d}{dx}$ (Dr) + c & proceed.

Example : Evaluate : $\int \frac{1}{1+\sin x+\cos x}\,dx$

Solution. $I = \int \frac{1}{1+\sin x+\cos x}\,dx$

$$= \int \frac{1}{1+\frac{2\tan x/2}{1+\tan^2 x/2}+\frac{1-\tan^2 x/2}{1+\tan^2 x/2}}\,dx$$

$$= \int \frac{1+\tan^2 x/2}{1+\tan^2 x/2 + 2\tan x/2 + 1 - \tan^2 x/2}\,dx = \int \frac{\sec^2 x/2}{2+2\tan x/2}\,dx$$

Putting $\tan\frac{x}{2} = t$ and $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$, we get

$$I = \int \frac{1}{t+1}\,dt = \log|t+1| + C = \log\left|\tan\frac{x}{2}+1\right| + C$$

Example : Evaluate : $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$

Solution. $I = \int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$

Let $3\sin x + 2\cos x = \lambda.\frac{d}{dx}(3\cos x + 2\sin x) + \mu(3\cos x + 2\sin x)$

$\Rightarrow$ $3\sin x + 2\cos x = \lambda(-3\sin x + 2\cos x) + \mu(3\cos x + 2\sin x)$

Comparing the coefficients of sin x and cos x on both sides, we get

$-3\lambda + 2\mu = 3$ and $2\lambda + 3\mu = 2$ $\Rightarrow$ $\mu = \frac{12}{13}$ and $\lambda = -\frac{5}{13}$

$$\therefore \quad I = \int \frac{\mu(-3\sin x + 2\cos x) + \lambda(3\cos x + 2\sin x)}{3\cos x + 2\sin x}\,dx$$

$$= \lambda\int 1.\,dx + \mu\int \frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$$

$= \lambda x + \mu \int \frac{dt}{t}$, where t = 3 cos x + 2 sin x

$= \lambda x + \mu \, \ell n \, | t | + C = \frac{-5}{13} x + \frac{12}{13} \ell n \, | 3 \cos x + 2 \sin x | + C$

Example : Evaluate : $\int \frac{3\cos x + 2}{\sin x + 2\cos x + 3} dx$

Solution. We have, $I = \int \frac{3\cos x + 2}{\sin x + 2\cos x + 3} dx$

Let 3 cos x + 2 = λ (sin x + 2 cos x + 3) + μ (cos x – 2 sin x) + ν
Comparing the coefficients of sin x, cos x and constant term on both sides, we get
λ – 2μ = 0, 2λ + μ = 3, 3λ + ν = 2

$\Rightarrow \quad \lambda = \frac{6}{5}, \mu \frac{3}{5}$ and $\nu = -\frac{8}{5}$

$\therefore \quad I = \int \frac{\lambda(\sin x + 2\cos x + 3) + \mu(\cos x - 2\sin x) + \nu}{\sin x + 2\cos x + 3} dx$

$\Rightarrow \quad I = \lambda \int dx + \mu \int \frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3} dx + \nu \int \frac{1}{\sin x + 2\cos x + 3} dx$

$\Rightarrow \quad = \lambda x + \mu \log | \sin x + 2 \cos x + 3 | + \nu I_1$, where

$I_1 = \int \frac{1}{\sin x + 2\cos x + 3} dx$

Putting, $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$, $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$ we get

$I_1 = \int \frac{1}{\frac{2\tan x/2}{1+\tan^2 x/2} + \frac{2(1-\tan^2 x/2)}{1+\tan^2 x/2} + 3} dx$

$= \int \frac{1+\tan^2 x/2}{2\tan x/2 + 2 - 2\tan^2 x/2 + 3(1+\tan^2 x/2)} dx$

$= \int \frac{\sec^2 x/2}{\tan^2 x/2 + 2\tan x/2 + 5} dx$

Putting $\tan \frac{x}{2} = t$ and $\frac{1}{2} \sec^2 \frac{x}{2} = dt$ or $\sec^2 \frac{x}{2} dx = 2\, dt$, we get

$I_1 = \int \frac{2dt}{t^2 + 2t + 5}$

$= 2 \int \frac{dt}{(t+1)^2 + 2^2} = \frac{2}{2} \tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{2}\right)$

Hence, $I = \lambda x + \mu \log | \sin x + 2 \cos x + 3 | + \nu \tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{2}\right) + C$

where $\lambda = \frac{6}{5}$, $\mu = \frac{3}{5}$ and $\nu = -\frac{8}{5}$

Example : $\int \frac{dx}{1+3\cos^2 x}$

Solution. $= \int \frac{\sec^2 x \, dx}{\tan^2 x + 4} \qquad = \frac{1}{2} \tan^{-1}\left(\frac{\tan x}{2}\right) + C$

Self Practice Problems

1. $\int \frac{4\sin x + 5\cos x}{5\sin x + 4\cos x} dx$ **Ans.** $\frac{40}{41} x + \frac{9}{41} \log |5\sin x + 4\cos x| + C$

10. Integration of type $\int \sin^m x . \cos^n x \, dx$

Case - I: If m and n are even natural number then converts higher power into higher angles.
Case - II: If at least m or n is odd natural number then if m is odd put cosx = t and vice-versa.
Case - III: When m + n is a negative even integer then put tan x = t.

Example: $\int \sin^5 x \cos^4 x \, dx$

Solution. put $\cos x = t \quad \Rightarrow \quad -\sin x \, dx = dt$

$= -\int (1-t^2)^2 \cdot t^4 \cdot dt \qquad = -\int (t^4 - 2t^2 + 1)\, t^4\, dt$

$= -\int (t^8 - 2t^6 + t^4)\, dt$

$= -\dfrac{t^9}{9} + \dfrac{2t^7}{7} - \dfrac{t^5}{5} + c$

$= -\dfrac{\cos^9 x}{9} + 2\,\dfrac{\cos^7 x}{7} - \dfrac{\cos^5 x}{5} + c$ **Ans.**

Example : $\int (\sin x)^{1/3} (\cos x)^{-7/3} dx$

Solution. $\int (\sin x)^{1/3} (\cos x)^{-7/3} dx$

$= \int (\tan x)^{1/3} \dfrac{1}{\cos^2 x}\, dx$

put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$= \int t^{1/3}\, dt \qquad = \dfrac{3}{4} t^{4/3} + c$

$= \dfrac{3}{4} (\tan x)^{4/3} + c$ **Ans.**

Example : $\int \sin^2 x \cos^4 x\, dx$

Solution. $\dfrac{1}{8} \int \sin^2 2x (1 + \cos 2x) dx$

$= \dfrac{1}{8} \int \sin^2 2x\, dx + \dfrac{1}{8} \int \sin^2 2x \cos 2x\, dx$

$= \dfrac{1}{16} \int (1 - \cos 4x)\, dx + \dfrac{1}{16} \left(\dfrac{\sin^3 2x}{3} \right)$

$= \dfrac{1}{16} - \dfrac{\sin 4x}{64} + \dfrac{\sin^3 2x}{48} + c$

11. Integration of type: $\int \dfrac{x^2 \pm 1}{x^4 + Kx^2 + 1} dx$ where K is any constant.

Divide Nr & Dr by x² & put $x \mp \dfrac{1}{x} = t$.

Example : $\int \dfrac{1 - x^2}{1 + x^2 + x^4} dx$

Solution. $\int \dfrac{\left(1 - \dfrac{1}{x^2}\right) dx}{x^2 + \dfrac{1}{x^2} + 1} \qquad x + \dfrac{1}{x} = t \qquad \Rightarrow \qquad -\int \dfrac{dt}{t^2 - 1}$

$-\dfrac{1}{2} \ell n \left| \dfrac{t-1}{t+1} \right| + C$

$-\dfrac{1}{2} \ell n \left| \dfrac{x + \dfrac{1}{x} - 1}{x + \dfrac{1}{x} + 1} \right| + C$

Example : Evaluate : $\int \dfrac{1}{x^4 + 1} dx$

Solution. We have,

$I = \int \dfrac{1}{x^4 + 1} dx$

$\Rightarrow \quad I = \int \dfrac{\dfrac{1}{x^2}}{x^2 + \dfrac{1}{x^2}} dx \qquad \Rightarrow \quad I = \dfrac{1}{2} \int \dfrac{\dfrac{2}{x^2}}{x^2 + \dfrac{1}{x^2}} dx$

$\Rightarrow \quad I = \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} - \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx \qquad \Rightarrow \quad I = \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$

$\Rightarrow \quad I = \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\,dx$

Putting $x - \frac{1}{x} = u$ in 1st integral and $x + \frac{1}{x} = \nu$ in 2nd integral, we get

$I = \frac{1}{2}\int \frac{du}{u^2+\left(\sqrt{2}\right)^2} - \frac{1}{2}\int \frac{d\nu}{\nu^2-\left(\sqrt{2}\right)^2}$

$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\,\frac{1}{2\sqrt{2}}\log\left|\frac{\nu-\sqrt{2}}{\nu+\sqrt{2}}\right| + C$

$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right| + C$

$= \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}\,x+1}{x^2+x\sqrt{2}+1}\right| + C$

Self Practice Problem :

1. $\int \frac{x^2-1}{x^4-7x^2+1}\,dx$ **Ans.** $\frac{1}{6}\,\ell n\left|\frac{x+\frac{1}{x}-3}{x+\frac{1}{x}+3}\right| + C$

2. $\int \sqrt{\tan x}\,dx$ **Ans.** $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\,\ell n\left|\frac{y-\sqrt{2}}{y+\sqrt{2}}\right|$ + C where $y = \tan x - \frac{1}{\tan x}$

12. Integration of type

$\int \frac{dx}{(ax+b)\sqrt{px+q}}$ **OR** $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$ **; put px + q = t².**

Example: Evaluate : $\int \frac{1}{(x-3)\sqrt{x+1}}\,dx$

Solution. Let $I = \int \frac{1}{(x-3)\sqrt{x+1}}\,dx$

Here, P and Q both are linear, so we put Q = t^2 i.e. x + 1 = t^2 and dx = 2t dt

$\therefore \quad I = \int \frac{1}{(t^2-1-3)}\,\frac{2t}{\sqrt{t^2}}\,dt$

$\Rightarrow \quad I = 2\int \frac{dt}{t^2-2^2} = 2\,.\,\frac{1}{2(2)}\log\left|\frac{t-2}{t+2}\right| + C \qquad \Rightarrow \quad I = \frac{1}{2}\log\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right| + C.$

Example : Evaluate : $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$

Solution. Let $I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$

Putting x + 1 = t^2, and dx = 2t dt, we get $I = \int \frac{(t^2+1)\,2t\,dt}{\{(t^2-1)^2+3(t^2-1)+3\}\sqrt{t^2}}$

$\Rightarrow \quad I = 2\int \frac{(t^2+1)}{t^4+t^2+1}\,dt = 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\,dt$

$\Rightarrow \quad I = 2\int \frac{du}{u^2+\left(\sqrt{3}\right)^2}$ where $t - \frac{1}{t} = u$. $\quad\Rightarrow \quad I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{t-\frac{1}{t}}{\sqrt{3}}\right\} + C$

$\Rightarrow \quad I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{t\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{x}{\sqrt{3(x+1)}}\right\} + C$

13. Integration of type

$$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}\text{, put } ax+b=\frac{1}{t}\text{; } \int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}\text{, put } x=\frac{1}{t}$$

Example : $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$

Solution $= \int \frac{-dt}{t^2\left(\frac{1}{t}\right)\sqrt{\left(\frac{1}{t}-1\right)^2+\frac{1}{t}}}$ $\qquad = \int \frac{-dt}{t\sqrt{\frac{1}{t^2}-\frac{1}{t}+1}}$

$= \int \frac{-dt}{\sqrt{t^2-t+1}}$ $\qquad = \int \frac{-dt}{\sqrt{\left(t-\frac{1}{2}\right)+\frac{3}{4}}}$

$= -\ell n\left(t-\frac{1}{2}+\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}\right) + C$

Example : $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Solution. Put $x = \frac{1}{t}$ $\Rightarrow$ $I = \int \frac{dt}{(t^2+1)\sqrt{t^2-1}}$

put $t^2 - 1 = y^2$

$\Rightarrow$ $I = -\int \frac{y\ dy}{(y^2+2)\,y}$ $\qquad = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + C$

$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + C$

Self Practice Problems :

1. $\int \frac{dx}{(x+2)\sqrt{x+1}}$ **Ans.** $2\tan^{-1}\left(\sqrt{x+1}\right) + C$

2. $\int \frac{dx}{(x^2+5x+6)\sqrt{x+1}}$ **Ans.** $2\tan^{-1}\left(\sqrt{x+1}\right)\ \sqrt{2}\tan^{-1}\left(\frac{\sqrt{x+1}}{\sqrt{2}}\right) + C$

3. $\int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$ **Ans.** $\sin^{-1}\left(\frac{\frac{3}{2}-\frac{1}{x+1}}{\frac{\sqrt{5}}{2}}\right) + C$

4. $\int \frac{dx}{(2x^2+1)\sqrt{1-x^2}}$ **Ans.** $-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1-x^2}{\sqrt{3}\,x^2}\right) + C$

5. $\int \frac{dx}{(x^2+2x+2)\sqrt{x^2+2x-4}}$ **Ans.** $-\frac{1}{2\sqrt{6}}\ell n\left(\frac{\sqrt{x^2+2x-4}-\sqrt{6}\,(x+1)}{\sqrt{x^2+2x-4}+\sqrt{6}\,(x+1)}\right) + C$

14. Integration of type

$\int\sqrt{\frac{x-\alpha}{\beta-x}}\,dx$ **or** $\int\sqrt{(x-\alpha)(\beta-x)}$; **put** $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int\sqrt{\frac{x-\alpha}{x-\beta}}\,dx$ **or** $\int\sqrt{(x-\alpha)(x-\beta)}$; **put** $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int\frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; **put** $x - \alpha = t^2$ **or** $x - \beta = t^2$.

15. Reduction formula of $\int \tan^n x\,dx$, $\int \cot^n x\,dx$, $\int \sec^n x\,dx$, $\int \operatorname{cosec}^n x\,dx$

1. $I_n = \int \tan^n x\,dx = \int \tan^2 x\tan^{n-2}x\,dx = \int(\sec^2 x-1)\tan^{n-2}x\,dx$

$\Rightarrow$ $I_n = \int \sec^2 x\tan^{n-2} + dx - I_{n-2}$ $\qquad\Rightarrow$ $I_n = \frac{\tan^{n-1}x}{n-1} - I_{n-2}$

2. $I_n = \int \cot^n x\,dx = \int \cot^2 .\cot^{n-2} x\,dx = \int (\text{cosec}^2 x - 1)\cot^{n-2} x\,dx$

$\Rightarrow \quad I_n = \int \text{cosec}^2 x \cot^{n-2} x\,dx - I_{n-2} \qquad \Rightarrow \quad I_n = -\dfrac{\cot^{n-1} x}{n-1} - I_{n-2}$

3. $I_n = \int \sec^n x\,dx = \int \sec^2 x \sec^{n-2} x\,dx$

$\Rightarrow \quad I_n = \tan x \sec^{n-2} x - \int (\tan x)(n-2) \sec^{n-3} x.\ \sec x \tan x\,dx.$

$\Rightarrow \quad I_n = \tan x \sec^{n-2} x\,dx - (n-2)(\sec^2 x - 1)\sec^{n-2} x\,dx$

$\Rightarrow \quad (n-1)\,I_n = \tan x \sec^{n-2} x + (n-2)\,I_n - 2$

$I_n = \dfrac{\tan x \sec^{n-2} x}{n-1} + \dfrac{n-2}{n-1}\,I_{n-2}$

4. $I_n = \int \text{cosec}^n dx = \int \text{cosec}^2 x\ \text{cosec}^{n-2} x\,dx$

$\Rightarrow \quad I_n = -\cot x\ \text{cosec}^{n-2} x + \int (\cot x)(n-2)\ (-\text{cosec}^{n-3} x\ \text{cosec}\,x \cot x)\,dx$

$\Rightarrow \quad -\cot x\ \text{cosec}^{n-2} x - (n-2) \int \cot^2 x\,\text{cosec}^{n-2} x\,dx$

$\Rightarrow \quad I_n = -\cot x\ \text{cosec}^{n-2} x - (n-2) \int (\text{cosec}^2 x - 1)\ \text{cosec}^{n-2} x\,dx$

$\Rightarrow \quad (n-1)\,I_n = -\cot x\ \text{cosec}^{n-2} x + (n-2)\,2_{n-2}$

$I_n = \dfrac{\cot x\,\text{cosec}^{n-2} x}{n-1} + \dfrac{n-2}{n-1}\,I_{n-2}$

Example : Obtain reducation formula for $I_n = \int \sin^n x\,dx$. Hence evaluate $\int \sin^4 x\,dx$

Solution. $I_n = \int \underset{\text{II}}{(\sin x)}\ \underset{\text{I}}{(\sin x)^{n-1}}\,dx$

$= -\cos x\,(\sin x)^{n-1} + (n-1) \int (\sin x)^{n-2} \cos^2 x\,dx$

$= -\cos x\,(\sin x)^{n-1} + (n-1) \int (\sin x)^{n-2}\,(1 - \sin^2 x)\,dx$

$I_n = -\cos x\,(\sin x)^{n-1} + (n-1)\,I_{n-2} - (n-1)\,I_n$

$\Rightarrow \quad I_n = -\dfrac{\cos x(\sin x)^{n-1}}{n} + \dfrac{(n-1)}{n}\,I_{n-2} \qquad (n \geq 2)$

Hence $I_4 = -\dfrac{\cos x(\sin x)^3}{4} + \dfrac{3}{4}\left(-\dfrac{\cos x(\sin x)}{2} + \dfrac{1}{2}x\right) + C$

Self Practice Problems :

1. $\int \sqrt{\dfrac{x-3}{x-4}}\,dx$ **Ans.** $\sqrt{(x-3)(x-4)} + \ell n\left(\sqrt{x-3} + \sqrt{x-4}\right) + C$

2. $\int \dfrac{dx}{[(x-1)(2-x)]^{3/2}}$ **Ans.** $8\left(\sqrt{\dfrac{x-1}{2-x}} - \sqrt{\dfrac{2-x}{x-1}}\right) + C$

3. $\int \dfrac{dx}{[(x+2)^8(x-1)^6]^{1/7}}$ **Ans.** $7\left(\dfrac{x-1}{x+2}\right)^{1/7} + C$

4. Deduce the reduction formula for $I_n = \int \dfrac{dx}{(1+x^4)^n}$ and Hence evaluate $I_2 = \int \dfrac{dx}{(1+x^4)^2}$

Ans. $I_n = \dfrac{x}{4(n-1)(1+x^4)^{n-1}} + \dfrac{4n-5}{4(n-1)}\,I_{n-1}$

$I_2 = \dfrac{x}{4\,(1+x^4)} + \dfrac{3}{4}\left(\dfrac{1}{2\sqrt{2}}\tan^{-1}\left(\dfrac{x - \frac{1}{x}}{\sqrt{2}}\right) - \dfrac{1}{4\sqrt{2}}\,\ell n\left(\dfrac{x + \frac{1}{x} - \sqrt{2}}{x + \frac{1}{x} + \sqrt{2}}\right)\right) + C$

5. If $I_{m,n} = \int (\sin x)^m (\cos x)^n\,dx$ then prove that

$I_{m,n} = \dfrac{(\sin x)^{m+1}(\cos x)^{n-1}}{m+n} + \dfrac{n-1}{m+n}\,.\,I_{m,n-2}$